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Transverse Waves in a Relativistic Rigid Body

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We present relativistic elasticity as a scalar field theory. We apply it to rigid bodies, i.e., relativistic bodies with a nonlinear elastic law and a definite longitudinal wave velocity v_l equal to the light velocity, c. We obtain the transverse wave equation with a definite velocity v_t and the relation between v_l, v_t, and the Poisson coefficient is the classical one. This is an indication that we have the relativistic extension of a classical Hooke elastic law.

A rigid body (Born, 1909), i.e., a body where the distance between any two points is constant, is a geometric abstraction with no physical existence in relativity. In fact, in such a body, the shock wave velocity is necessarily infinite. However, in the framework of special relativity, one can define (Brotas, 1969a) a rigid body as one where longitudinal shock waves propagate with the speed of light c. The elastic law of such a rigid body was first derived for a one-dimensional rod (McCrea, 1952; McCrea and Hogarth, 1952) and latter generalized for a three-dimensional body with zero Poisson coefficient (Brotas, 1969b). Our aim in this note is to derive the equations of motion of this so-called rigid body in order to study the propagation of transverse waves.

Let x^μ ($\mu = 0, 1, 2, 3$) be the Cartesian coordinates of an inertial frame S in the Minkowski space. We label the body "particles" with the coordinates $\bar{x}^i$ ($i = 1, 2, 3$). One can describe the motion by the equations $\bar{x}^i = \bar{x}^i(x^\mu)$; one can regard the $\bar{x}^i$ as scalar fields because their values are independent of the coordinate system.[2] Assuming that the equations of adiabatic motion can be derived in a variational way from an appropriate Lagrangian density $\mathscr{L}$, function of the fields and their first derivatives, the equations of motion

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[2]We use the summation convention for all indices, with $c = 1$, and the metric signature is $(+, -, -, -)$.

are

$$\frac{\partial \mathscr{L}}{\partial \bar{x}^i} = \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \bar{x}^i)} \tag{1}$$

and the energy–momentum tensor, $T^{\mu\nu}$, is

$$T^{\mu\nu} = -g^{\mu\nu}\mathscr{L} + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \bar{x}^i)} \partial^\nu_{\bar{x}^i} \tag{2}$$

Our problem here is to find the Lagrangian density which corresponds to the rigid body. Clearly, in a coordinate system x_0^μ, associated with the local frame S_0, where the material point $\bar{P}(\bar{x}^i)$ is instantaneously at rest ($\partial_0 \bar{x}^i = 0$), the energy density T^{00} is equal to the elastic energy density ρ_0 since there is no kinetic contribution. Hence, we have

$$\rho_0 = -\mathscr{L} \tag{3}$$

In the coordinate system $y^\mu \equiv (x_0^0, \bar{x}^i)$ the metric tensor $g^{\mu\nu}$ is such that $g_{0i} = 0$, i.e.,

$$ds^2 = (dy^0)^2 - dl_0^2 \tag{4}$$

with

$$dl_0^2 = \gamma_{ij}\, d\bar{x}^i\, d\bar{x}^j \tag{5}$$

Since, dl_0 is the spatial distance in the proper frame S_0, the set of scalar coefficients γ^{ij} [elements of the inverse matrix of (γ_{ij})],

$$\gamma^{ij} = -\partial^\mu \bar{x}^i\, \partial_\mu \bar{x}^j \tag{6}$$

characterize the body deformation. The elastic energy density ρ_0 can only depend on the point of the body specified by $\bar{x}^k$ and on the state of strain which is given by γ^{ij}. So, equations (3) and (6) show that it is possible to define $\mathscr{L}$ as a function of the fields $\bar{x}^i$ and their derivatives. Note that $\partial^\alpha \bar{x}^i$ are spacelike 4-vectors orthognal to the body velocity $u^\alpha \equiv (\partial x^\alpha/\partial y^0)_{\bar{x}^i}$. The Born rigid motion is given by $u^\alpha \partial_\alpha \gamma_{ij} = 0$. On the other hand, for our rigid body, equations (1)-(3) and (6) give

$$\frac{1}{2}\frac{\partial \rho_0}{\partial \bar{x}^i} + \partial_\mu\left(\frac{\partial \rho_0}{\partial \gamma^{ij}}\, \partial^\mu \bar{x}^j\right) = 0 \tag{7}$$

$$T^{\mu\nu} = g^{\mu\nu}\rho_0 + 2\frac{\partial \rho_0}{\partial \gamma^{ij}}\, \partial^\mu \bar{x}^i\, \partial^\nu \bar{x}^j \tag{8}$$

where we have assumed, without loss of generality, that $\partial\rho_0/\partial\gamma^{ij} = \partial\rho_0/\partial\gamma^{ji}$.

In order to write ρ_0 as an explicit function of γ^{ij} we consider the particular class of coordinates $\bar{x}^i$ such that $\gamma^{ij} = \delta_{ij}$ whenever there is no deformation. Hence, the classical strain tensor e_{ij} is

$$e_{ij} = \tfrac{1}{2}(\gamma_{ij} - \delta_{ij}) \tag{9}$$

In the stationary situation, where $\bar{x}^i = (\Gamma^i)^{1/2} x^i$ $(i = 1, 2, 3)$ with Γ^i constants, equation (6) shows that the matrix $\gamma = (\gamma^{ij})$ is diagonal and $\gamma^{ii} = \Gamma^i$. The Brotas rigid solid, S, with mass density $\overset{0}{\rho_0}(\bar{x}^k)$ has an energy density $\overset{s}{\rho_0}$ given by (Brotas, 1969b)

$$\overset{s}{\rho_0} = \frac{\overset{0}{\rho_0}}{8}(\Gamma^1 + 1)(\Gamma^2 + 1)(\Gamma^3 + 1) \tag{10}$$

and a diagonal stress tensor with

$$T_s^{11} = \frac{\overset{0}{\rho_0}}{8}(\Gamma^1 - 1)(\Gamma^2 + 1)(\Gamma^3 + 1) \tag{11}$$

and the other T_s^{ii} are obtained from T_s^{11} by permutation of Γ^i and Γ^1. We observe that equation (10) can be written as

$$\overset{s}{\rho_0} = \frac{\overset{0}{\rho_0}}{8}(\lambda_0 + \lambda_1 + \lambda_2 + 1) \tag{12}$$

with

$$\begin{aligned} \lambda_0 &= \det \gamma \\ \lambda_1 &= \operatorname{tr} \gamma \\ \lambda_2 &= A^{11} + A^{22} + A^{33} \end{aligned} \tag{13}$$

where A^{ij} are the γ cofactors. Since we are dealing with the class of coordinates $\bar{x}^i$ such that $e_{ij} = 0$ is an invariant equation, the expression of ρ_0 as a function of λ_0, λ_1, λ_2 is also invariant. This is true because any two coordinate systems of this class are necessarily related by an orthogonal transformation, under which the λ_i are invariants. These transformations represent rotations, inversions, or permutations of the axis in the $\bar{x}^i$ space. So the elastic law (12) is clearly isotropic, since the functional dependence of $\overset{s}{\rho_0}$ on γ^{ij} does not depend on the choice of the body axis. Reciprocally, for any isotropic law, since γ can be diagonalized, ρ_0 is a symmetric function of the γ eigenvalues, Γ^i, which are determined by the λ_j. Thus we have shown that for any isotropic elastic law ρ_0 depends on γ through the invariants λ_0, λ_1, λ_2.

Using the elastic law (12) and equation (8) we derived the diagonalized stress tensor given by (11). On the other hand from the equations of motion (7) we obtained the velocity of the transverse waves $v_t = c/\sqrt{2}$. We observe that the relation between v_t and the longitudinal wave velocity, $v_l = c$, is characteristic of a solid with zero Poisson coefficient, σ. One can idealize[3] a solid with arbitrary σ as a point by point superposition and binding of a pure solid ($\sigma = 0$) with density $(1-\alpha)\rho_0^0$ and a liquid with density $\alpha\rho_0^0$ ($0 \leq \alpha \leq 1$). Naturally, we try the elastic law

$$\rho_0 = (1-\alpha)\rho_0^s + \alpha\rho_0^l \tag{14}$$

where ρ_0^s and ρ_0^l are the energy densities of the pure rigid solid and the rigid liquid, respectively. For the latter we use

$$\rho_0^L = \frac{\rho_0^0}{2}(\lambda_0 + 1) \tag{15}$$

which is a generalization of McCrea's one-dimensional law

$$\rho_0 = \frac{\rho_0^0}{2}(\gamma^{11} + 1) \tag{16}$$

In fact, in the one-dimensional case the proper length dl_0 is

$$dl_0 = (\gamma^{11})^{-1/2}\, d\bar{x}$$

whereas in the liquid, or solid, the proper volume is $\lambda_0^{-1/2}\, d\bar{x}\, d\bar{y}\, d\bar{z}$. Hence, the replacement $\gamma^{11} \to \lambda_0$ leads to the liquid law.

Let us now study the propagation of transverse waves in a material with an elastic law given by equation (14). For simplicity, we shall consider a homogeneous two-dimensional material, i.e., ρ_0^0 is constant and $\bar{z} = z$. Then equation (7) lead to

$$\partial_\mu\left[\left(\frac{1-\alpha}{1+\alpha} + \gamma^{22}\right)\partial^\mu \bar{x} - \gamma^{12}\,\partial^\mu \bar{y}\right] = 0 \tag{17$\bar{x}$}$$

$$\partial_\mu\left[\left(\frac{1-\alpha}{1+\alpha} + \gamma^{11}\right)\partial^\mu \bar{y} - \gamma^{12}\,\partial^\mu \bar{x}\right] = 0 \tag{17$\bar{y}$}$$

where the first equation refers to the field $\bar{x}$ and the second one to $\bar{y}$. Clearly, if in these equations we make $\bar{y} = y$ we obtain $\partial_\mu\, \partial^\mu \bar{x} = 0$, which corresponds to a longitudinal wave velocity equal to c. For a transversal signal ε propagating along the xx axis, we have $\bar{x} = x$, $\bar{y} = y - \varepsilon(t, x)$ and the previous equations give

$$-\frac{\partial\varepsilon}{\partial t}\frac{\partial^2\varepsilon}{\partial t\,\partial x} + \frac{\partial\varepsilon}{\partial x}\frac{\partial^2\varepsilon}{\partial t^2} = 0 \tag{18$\bar{x}$}$$

$$\frac{\partial^2\varepsilon}{\partial t^2} - \frac{1-\alpha}{2}\frac{\partial^2\varepsilon}{\partial x^2} = 0 \tag{18$\bar{y}$}$$

[3]The author is indebted to A. Brotas for explicitly pointing out this idea.

Obviously, any function $\varepsilon = f(x \pm v_t t)$ with

$$v_t = \left(\frac{1-\alpha}{2}\right)^{1/2} c \tag{19}$$

is a solution of equation ($18\bar{y}$). Although equation ($18\bar{x}$) is not linear it is easy to see that it has the same solution. The nonlinearity of equation ($18\bar{x}$) invalidates the superposition principle. This is a relativistic feature. Linearizing the expression of the stationary stress tensor, $T = (1-\alpha)T_s + \alpha T_L$ [cf. equations (11) and (15)], one obtains for the Poisson coefficient σ,

$$\sigma = \frac{\alpha}{1+\alpha} \tag{20}$$

It is interesting to point out that equation (20) implies the classical range of variation for σ, $0 \leq \sigma \leq 1/2$. Using this result in equation (19) we obtain

$$(v_t/v_l)^2 = \frac{1-2\sigma}{2(1-\sigma)} \tag{21}$$

which is a well-known classical result. This we regard as an indication that the nonlinear elastic law (14) is the relativistic extension of an Hooke law.

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